Geometric Invariance

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October 17, 2003

Much of differential geometry is devoted to finding intrinsic properties of curves, surfaces and higher dimensional manifolds. An intrinsic property is one that does not depend on the parametrization or embedding of the manifold.

Intrinsic properties are invariants; curvature is invariant under translation, for example. We want to find objects invariant under reparametrization, that is, an objects that is the same for two parametrizations of the same set of points.

Example 1

Consider the mappings $t \to (\cos t, 0, \sin t)$ and $s \to (cos(-s), 0, sin(-s))$. Both are parametrizations of the circle. This example shows that T, N, and B are *not* invariants.

We can also study objects invariant under Euclidean motion. T, N, and B also fail to be invariants of this type.

Theorem 1: τ, κ and s are invariant under reperamtetrization of a given curve.

To prove this theorem we need a tool to relate two parametrizations of a curve.

Def: Suppose $\alpha : [a, b] \to \mathbf{R}^3$ and $\beta : [c, d] \to \mathbf{R}^3$ are two curves. Then α is equivalent to β, or α and β are two parametrizations of the same curve if $\exists$ $\phi : [a, b] \to [c, d]$ such that:

(1) $\phi' \neq 0$
(2) $\phi(a) = c$ and $\phi(b) = d$ or $\phi(a) = d$ and $\phi(b) = c$
(3) ϕ is suitably differentiable, say C^3
(4) $\beta \circ \phi = \alpha$.

In practice, it's not hard to find ϕ, it is $\beta^{-1} \circ \alpha$. Then $\phi' \neq 0$ follows from the non-vanishing of β' and α'. Differentiability of ϕ also follows from that of α and β. Note that (1) implies ϕ is a strictly increasing or decreasing function.

Example 2: $\alpha : [0, 2\pi] \to \mathbf{R}^3$ by $\alpha(t) = (\cos t, \sin t, 0)$ and $\beta : [0, \pi] \to \mathbf{R}^3$

by $\beta(u) = (\cos 2u, \sin 2u, 0)$. Then ϕ is given by $p \mapsto \frac{p}{2} : \beta(\phi(t)) = \beta(\frac{t}{2}) = (\cos t, \sin t, 0) = \alpha(t)$. Equivalently, $u = \frac{t}{2}$.

Example 3: $\alpha : [0, 2\pi] \to \mathbf{R}^3$, $\alpha(t) = (\cos t, \sin t, 0)$. $\beta : [0, 2\pi] \to \mathbf{R}^3$, $\beta(u) = (\cos u, 0, \sin u)$. Then $\beta^{-1} \circ \alpha$ cannot be defined, so according to our definition, these curves are not equivalent.

1 Independence Under Parametrization of Arclength

Let α, β be two parametrizations: $\alpha : [a, b] \to \mathbf{R}^3, \beta : [c, d] \to \mathbf{R}^3$. WLOG, let $\alpha(a) = c$, $\alpha(b) = d$ and $\beta \circ \phi = \alpha$. Then the arclength L of the curve is given by:

$$L = \int_a^b ||\alpha'(t)||dt = \int_a^b ||(\beta \circ \phi)'(t)||dt = \int_a^b ||\beta'(\phi(t))\phi'(t)||dt$$

Now let $\phi(t) = u$. Then

$$L = \int_a^b ||\beta'(\phi(t))|| \cdot ||\phi'(t)||dt = \int_c^d ||\beta'(u)||du.$$

We're justified in using the Change of Variables Theorem since $\phi' \neq 0$.

2 Curvature

Last class we showed the invariance of curvature as follows. Let α, β and ϕ be as above with $\alpha(t) = \beta(\phi(t))$. Set $\phi(t) = u$. Let's calculate first and second derivatives of α.

$$\alpha'(t) = \beta'(\phi(t)) \cdot \phi'(t)$$

$$\alpha''(t) = \beta''(\phi(t)) \cdot \phi'(t)^2 + \beta'(\phi(t))\phi''(t).$$

Now, let's use this information to show that the curvature of the curve is independent of parametrization. That is, we'll show the curvature is the same whether we use α or β as our map.

$$\kappa(t) = \left.\frac{||\alpha' \times \alpha''||}{||\alpha'||}\right|_t = \left.\frac{||(\beta' \circ \phi) \cdot \phi' \times [(\beta'' \circ \phi) \cdot \phi'^2 + (\beta' \circ \phi) \cdot \phi'']||}{||\beta' \circ \phi||^3 \cdot ||\phi'||^3}\right|_t$$

$$= \left.\frac{||(\beta' \circ \phi) \cdot \phi' \times (\beta'' \circ \phi) \cdot \phi'^2||}{||\beta' \circ \phi||^3 \cdot ||\phi'||^3}\right|_t$$

$$= \left.\frac{||(\beta' \circ \phi) \times (\beta'' \circ \phi)||}{||\beta' \circ \phi||^3}\right|_t$$

$$= \frac{||\beta' \times \beta''||}{||\beta'||^3}\bigg|_{\phi(t)}$$

For planar curves, however, there's a subtlety since κ may be negative. In this case, it's true that $|\kappa|$ will be invariant and we can gain information depending on whether the curvatures of two parametrizations have the same or opposite signs. Specifically, they have the same sign if and only if $\phi' > 0$ and opposite signs if and only if $\phi' < 0$.

Example 4: $t \to (\cos t, \sin t)$ and $u \to (\cos(-u), \sin(-u))$ where $t, u \in [0, 2\pi]$ are two parametrizations of the same figure, namely a circle. In this case, $\phi' \equiv -1$ so the parametrizations have opposite orientations. This makes sense: as t increases, we move around the circle counter-clockwise, ad as u increases, we move around the circle clockwise. The proof of the invariance of torsion is similar.

3 Surface Invariants

The definition of a reparametrization of a surface is the natural generalization of that for a curve.

Def: Suppose $x : [a, b] \times [c, d] \to \mathbf{R}^3$ and $y : [e, f] \times [g, h] \to \mathbf{R}^3$ are two surfaces embedded in $\mathbf{R}^3$. x and y are *equivalent* if $\exists$ a C^3 function $\Phi : [a, b] \times [c, d] \to [e, f] \times [g, h]$ such that $y \circ \Phi = x$ and $det(J\Phi(u, v)) \neq 0 \quad \forall (u, v) \in [a, b] \times [c, d]$. Here

$$J\Phi = \begin{pmatrix} \frac{d\Phi_1}{du} & \frac{d\Phi_1}{dv} \\ \frac{d\Phi_2}{du} & \frac{d\Phi_2}{dv} \end{pmatrix}$$

– is the standard Jacobian matrix.

Theorem: Surface area is independent of parametrization.

Proof: Let x, y and Φ be as in the definition above. Then

$$SA = \int_c^d \int_a^b ||x_u \times x_v|| du\, dv.$$

Let $\Phi_1(u, v) = \hat{u}$ and $\Phi_2(u, v) = \hat{v}$.

Lemma: $||x_u \times x_v|| = |J\Phi| \cdot ||y_{\hat{u}} \times y_{\hat{v}}||$

Proof of Lemma:

$$||x_u \times x_v|| = ||\frac{d}{du}(y \circ \Phi) \times \frac{d}{dy}(y \circ \Phi)||$$

$$= ||(\frac{dy}{d\Phi_1}\frac{d\Phi_1}{du} + \frac{dy}{d\Phi_2}\frac{d\Phi_2}{du}) \times (\frac{dy}{d\Phi_1}\frac{d\Phi_1}{dv} + \frac{dy}{d\Phi_2}\frac{d\Phi_2}{dv})||$$

$$= ||\frac{d\Phi_1}{du}\frac{d\Phi_1}{dv}(\frac{dy}{d\Phi_1}\times\frac{dy}{d\Phi_1})+\frac{d\Phi_1}{du}\frac{d\Phi_2}{dv}(\frac{dy}{d\Phi_1}\times\frac{dy}{d\Phi_2})+\frac{d\Phi_2}{du}\frac{d\Phi_1}{dv}(\frac{dy}{d\Phi_2}\times\frac{dy}{d\Phi_1})+\frac{d\Phi_2}{du}\frac{d\Phi_2}{dv}(\frac{dy}{d\Phi_2}\times\frac{dy}{d\Phi_2})||$$

$$= ||\frac{d\Phi_1}{du}\frac{d\Phi_2}{dv}-\frac{d\Phi_2}{du}\frac{d\Phi_1}{dv}||\cdot||\frac{dy}{d\Phi_1}\times\frac{dy}{d\Phi_2}||$$

$$= |J\Phi|\cdot||y_{\hat{u}}\times y_{\hat{v}}||$$

With this lemma proved, the proof of the Theorem is a cinch.

Proof of Theorem:

$$SA(x) = \int_c^d\int_a^b ||x_u\times x_v||du\,dv$$

$$= \int_c^d\int_a^b |J\Phi|\cdot||y_{\hat{u}}\times y_{\hat{v}}||du\,dv$$

Where we've used the Lemma. Then by the Change of Variables Theorem,

$$= \int_g^h\int_e^f ||y_{\hat{u}}\times y_{\hat{v}}||d\hat{u}\,d\hat{v} = SA(y).$$

Note 1) This is the same technique used to show invariance of path-length.

Note 2) We've shown $\sqrt{EG-F^2}\cdot|J\Phi| = \sqrt{\hat{E}\hat{G}-\hat{F}^2}$.

Theorem: Gauss curvature in invariant under reparametrization.

Proof: You do it!

Hint: Try to prove something similar to Note 2) for $LN - M^2$
These results are good, but still not good enough. In particular, we've only shown invariance for curves and surfaces in $\mathbf{R}^3$. These objects may very well be embedded in higher dimensions.

4 Invariance Under Euclidean Motion

It's intuitively obvious that rotating or translating a curve will not effect s, κ, or τ, but here's how to prove it.

Theorem: s, κ and τ are invariant under translation.

Proof: Let $x(t)$ be a suitably differentiable curve. Let $y(t) = x(t) + \vec{v}$, $\vec{v}$ a constant vector. Then $x' = y'\, x'' = y'', and\, x''' = y'''$. Since s, κ and τ are expressed in terms of derivatives, the theorem is proved.

To prove that arclength is preserved by rotation, note that any rotation in

$\mathbf{R}^3$ is given by an orthogonal matrix O. Since arclength depends on $||x'(t)||$, it suffices to show that O preserves lengths:

$$||Ov||^2 =< Ov, Ov >=< v, O^T Ov >=< v, O^{-1}Ov >=< v, v >= ||v||^2.$$

Now let's show that curvature is invariant under rotation. WLOG, let x be parametrized with respect to arclength. Let O be a rotation matrix and let $y = O \circ x$. Then $y' = \frac{dO}{dx}\frac{dx}{ds} = O(x'(s))$ since O is linear. Similarly, $\frac{d^2}{ds^2}O(x(s)) = O(x''(s))$. So $||y' \times y''|| = ||O(x'(s)) \times O(x''(s))|| = ||x' \times x''||$. This is because orthogonal matrices preserve norms of cross products.